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INFLUENCE OF A SMALL PERTURBATION ON POINCARÉ-ANDRONOV OPERATORS WITH NOT WELL DEFINED TOPOLOGICAL DEGREE

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(*Submitted by J. Mawhin*)

Abstract. Let $\mathcal{P}_\varepsilon \in C^0(\mathbf{R}^n, \mathbf{R}^n)$ be the Poincaré-Andronov operator over period $T > 0$ of T-periodically perturbed autonomous system $\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$, where $\varepsilon > 0$ is small. Assuming that for $\varepsilon = 0$ this system has a T-periodic limit cycle x_0 we evaluate the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ of $I - \mathcal{P}_\varepsilon$ on an open bounded set U whose boundary ∂U contains $x_0([0, T])$ and $\mathcal{P}_0(v) \neq v$ for any $v \in \partial U \backslash x_0([0, T])$. We give an explicit formula connecting $d(I - \mathcal{P}_\varepsilon, U)$ with the topological indices of zeros of the associated Malkin's bifurcation function. The goal of the paper is to prove the Mawhin's conjecture claiming that $d(I - \mathcal{P}_\varepsilon, U)$ can be any integer in spite of the fact that the measure of the set of fixed points of $\mathcal{P}_0$ on ∂U is zero.

1. Introduction

Consider the system of ordinary differential equations

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \tag{1.1}$$

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where $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$, $g \in C^0(\mathbf{R} \times \mathbf{R}^n \times [0,1], \mathbf{R}^n)$, $g(t+T, v, \varepsilon) \equiv g(t, v, \varepsilon)$ and $\varepsilon > 0$ is a small parameter. We suppose that equation (1.1) defines a flow in $\mathbf{R}^n$, i.e. assume the uniqueness and global existence for the solutions of the Cauchy problems associated to (1.1). For each $v \in \mathbf{R}^n$ we denote by $x_\varepsilon(\cdot, v)$ the solution of (1.1) with $x_\varepsilon(0, v) = v$. Thus, the Poincaré-Andronov operator over the period $T > 0$ is defined by

$$\mathcal{P}_\varepsilon(v) := x_\varepsilon(T, v).$$

The problem of the existence (and even stability, see Ortega [11]) of T-periodic solutions of (1.1) with initial conditions inside an open bounded set U can be solved by evaluating the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ of $I - \mathcal{P}_\varepsilon$ on U (see [6]). In the case when $\mathcal{P}_0$ has no fixed points on the boundary ∂U of U the problem is completely solved by Capietto, Mawhin and Zanolin [2] who proved that $d(I - \mathcal{P}_0, U) = (-1)^n d(f, U)$ generalizing the result by Berstein and Halanay [1] where U is assumed to be a neighborhood of an isolated zero of f. In the case when $\mathcal{P}_0$ has fixed points on ∂U the pioneer result has been obtained by Mawhin [10] who considered the situation when $f = 0$. Mawhin proved that if $g_0(v) = \int_0^T g(\tau, v, 0) d\tau$ does not vanish on ∂U then $d(I - \mathcal{P}_\varepsilon, U)$ is defined for $\varepsilon > 0$ sufficiently small and it can be evaluated as $d(I - \mathcal{P}_\varepsilon, U) = d(-g_0, U)$. This paper studies an intermediate situation when the fixed points of $\mathcal{P}_0$ fill a part of ∂U. Current results on this subject deal with the case when ∂U contains a fixed number of fixed points, e.g. Feckan [4], Kamenskii-Makarenkov-Nistri [5]. As a part of a wider study of this problem Jean Mawhin (his seminar, November 2005) asked a question on evaluating $d(I - \mathcal{P}_\varepsilon, U)$ in the case when ∂U contains a curve of fixed points of $\mathcal{P}_0$. He settled the following conjecture:

Mawhin's conjecture. *For small $\varepsilon > 0$ the topological degree $d(I - \mathcal{P}_\varepsilon, U)$ can be any integer depending on the perturbation term g in spite of the fact that the measure of $\{v \in \partial U : \mathcal{P}_0(v) = v\}$ is zero.*

The goal of this paper is to evaluate $d(I - \mathcal{P}_\varepsilon, U)$ and to give a proof of the above conjecture in the case when $\{v \in \partial U : \mathcal{P}_0(v) = v\}$ forms a curve coming from a T-periodic limit cycle of the unperturbed system

$$\dot{x} = f(x). \tag{1.2}$$

Our fundamental assumption is that the algebraic multiplicity of the multiplicator $+1$ of the linearized system

$$\dot{y} = f'(x_0(t))y \tag{1.3}$$

equals to 1. In this case we say that the cycle x_0 is nondegenerate.

The paper is organized as follows. In Section 2 for a fixed point v_ε of $\mathcal{P}_\varepsilon$ satisfying $v_\varepsilon \to v_0 \in x_0([0,T])$ as $\varepsilon \to 0$ we obtain an asymptotic direction of the vector $v_\varepsilon - v_0$. By means of this result we evaluate in Section 3 the topological index of such fixed points $v_\varepsilon \to v_0 \in x_0([0,T])$ as $\varepsilon \to 0$ that $v_\varepsilon \in U$. Finally in Section 4 we give a proof of the Mawhin's conjecture provided that a technical assumption (see assumption 4.1) is satisfied.

2. Direction the fixed points of Poincaré-Andronov operator move when the perturbation increases

Since the cycle x_0 is nondegenerate we can define (see [3], Ch. IV, § 20, Lemma 1) a matrix function Z_{n-1} solving the adjoint system

$$\dot{z} = -(f'(x_0(t)))^* z \tag{2.1}$$

and having the form $Z_{n-1}(t) = \Phi(t)\mathrm{e}^{\Lambda t}$, where Φ is a continuous T-periodic $n \times n-1$ matrix function and Λ is a $n-1 \times n-1$-matrix with different from 0 eigenvalues. Let z_0 be the T-periodic solution of (2.1) satisfying $z_0(0)^*\dot{x}_0(0) = 1$. Finally, we denote by Y_{n-1} the $n \times n-1$ matrix function whose columns are solutions of the linearized system (1.3) satisfying $Y_{n-1}(0)^* Z_{n-1}(0) = I$.

The results of this paper are formulated in terms of the following auxiliary functions:

$$M(\theta) = \int_0^T z_0(\tau)^* g(\tau - \theta, x_0(\tau), 0)d\tau,$$

$$M^\perp(t,\theta) = \left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \int_{t-T+\theta}^{t+\theta} (Z_{n-1}(\tau))^* g(\tau-\theta, x_0(\tau), 0)d\tau,$$

$$\angle(u,v) = \arccos \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}.$$

The function M was proposed by Malkin (see [9], formula 3.13) and the function $M^\perp$ is a generalization of the function $M_z^\perp$ of [8].

Next Theorem 2.1 shows that if a family $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ of T-periodic solutions of (1.1) emanate from $x_0(\cdot + \theta_0)$ then a suitable projection of $x_{\varepsilon,\lambda}(t) - x_0(t + \theta_0)$ can be always controlled. Though motivated by the Mawhin's conjecture, Theorem 2.1 can be of a general interest in the theory of oscillations playing a role of the first approximation formula (see Loud [7], formula 1.3, Lemma 1 and formula for x at p. 510) in the case when the zeros of the bifurcation function M are not necessary isolated.

THEOREM 2.1. *Let x_0 be a nondegenerate T-periodic cycle of (1.2). Let $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ be a family of T-periodic solutions of (1.1) such that $x_{\varepsilon,\lambda}(t)\to x_0(t+\theta_0)$ as $\varepsilon\to 0$ uniformly with respect to $t\in[0,T]$ and $\lambda\in\Lambda$. Then*

$$\angle\left(Z_{n-1}(t+\theta_0)^*(x_{\varepsilon,\lambda}(t)-x_0(t+\theta_0)),M^{\perp}(t,\theta_0)\right)\to 0 \quad \textit{as } \varepsilon\to 0$$

uniformly with respect to $t\in[0,T]$ and $\lambda\in\Lambda$.

Proof. The proof makes use of the idea of Theorem 3.1 of [8]. In the sequel (A,B) denotes the matrix composed by columns of matrixes A and B. Let $a_\varepsilon\in C^0([0,T],\mathbf{R}^n)$ be given by

$$a_\varepsilon(t)=(z_0(t+\theta_0),Z_{n-1}(t+\theta_0))^*\,(x_\varepsilon(t)-x_0(t+\theta_0)). \tag{2.2}$$

Denoting $Y(t)=(\dot x_0(t),Y_{n-1}(t))$ by Perron's lemma [12] (see also Demidovich ([3], Sec. III, §12) we have

$$(z_0(t),Z_{n-1}(t))^*\,Y(t)=I, \quad \text{for any } t\in\mathbf{R}.$$

Thus

$$x_\varepsilon(t)-x_0(t+\theta_0)=Y(t+\theta_0)a_\varepsilon(t), \quad \text{for any } t\in\mathbf{R}. \tag{2.3}$$

By subtracting (1.2) where x is replaced by $x_0(\cdot+\theta_0)$ from (1.1) where x is replaced by x_ε we obtain

$$\dot x_\varepsilon(t)-\dot x_0(t+\theta_0)=f'(x_0(t+\theta_0))(x_\varepsilon(t)-x_0(t+\theta_0))$$

$$+\varepsilon g(t,x_\varepsilon(t),\varepsilon)+o(t,x_\varepsilon(t)-x_0(t+\theta_0)), \tag{2.4}$$

where $o(t,v)/\|v\|\to 0$ as $\mathbf{R}^n\ni v\to 0$ uniformly with respect to $t\in[0,T]$. By substituting (2.3) into (2.4) we have

$$\dot Y(t+\theta_0)a_\varepsilon(t)+Y(t+\theta_0)\dot a_\varepsilon(t)$$

$$=f'(x_0(t+\theta_0))Y(t+\theta_0)a_\varepsilon(t)+\varepsilon g(t,x_\varepsilon(t),\varepsilon)+o(t,x_\varepsilon(t)-x_0(t+\theta_0)).$$

Since $f'(x_0(t))Y(t)=\dot Y(t)$ the last relation can be rewritten as

$$Y(t+\theta_0)\dot a_\varepsilon(t)=\varepsilon g(t,x_\varepsilon(t),\varepsilon)+o(t,x_\varepsilon(t)-x_0(t+\theta_0)). \tag{2.5}$$

Applying $Z_{n-1}(t+\theta_0)^*$ to both sides of (2.5) we have

$$(0,I)\dot a_\varepsilon(t)=\varepsilon Z_{n-1}(t+\theta_0)^*\,g(t,x_\varepsilon(t),\varepsilon)+Z_{n-1}(t+\theta_0)^*\,o(t,x_\varepsilon(t)-x_0(t+\theta_0)),$$

where 0 denotes the $n-1$ dimensional zero vector and I stays for the identical $n-1\times n-1$ matrix. So

$$(0,I)a_\varepsilon(t)=(0,I)a_\varepsilon(t_0)+\varepsilon\int\limits_{t_0}^{t}Z_{n-1}(\tau+\theta_0)^*g(\tau,x_\varepsilon(\tau),\varepsilon)d\tau$$

$$+\int_{t_0}^{t} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau)-x_0(\tau+\theta_0))d\tau. \tag{2.6}$$

From the definition of Z_{n-1} we have that $Z_{n-1}(t)^* = \left(e^{\Lambda T}\right)^* Z_{n-1}(t-T)^*$ for any $t \in \mathbf{R}$ and so $(0,I)a_\varepsilon(t)$ satisfies

$$(0,I)a_\varepsilon(t_0) = \left(e^{\Lambda T}\right)^* (0,I)a_\varepsilon(t_0-T) \quad \text{for any } t_0 \in [0,T]. \tag{2.7}$$

Solving (2.6)-(2.7) with respect to $(0,I)a_{\varepsilon,n}(t_0)$ we obtain

$$(0,I)a_\varepsilon(t_0) = \varepsilon \left(e^{\Lambda T}\right)^* \left(\left(e^{\Lambda T}\right)^* - I\right)^{-1} \int_{t_0-T}^{t_0} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau), \varepsilon)d\tau$$

$$+\left(e^{\Lambda T}\right)^* \left(\left(e^{\Lambda T}\right)^* - I\right)^{-1} \int_{t_0-T}^{t_0} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau)-x_0(\tau+\theta_0))d\tau$$

for any $t_0 \in [0,T]$. On the other hand from (2.2) we obtain

$$Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)) = (0,I)a_\varepsilon(t)$$

and therefore

$$Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)) - q_\varepsilon(t)$$

$$= \varepsilon \left(e^{\Lambda T}\right)^* \left(\left(e^{\Lambda T}\right)^* - I\right)^{-1} \int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau), \varepsilon)d\tau, \tag{2.8}$$

where

$$q_\varepsilon = \left(e^{\Lambda T}\right)^* \left(\left(e^{\Lambda T}\right)^* - I\right)^{-1} \int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* o(\tau, x_\varepsilon(\tau)-x_0(\tau+\theta_0))d\tau.$$

From (2.8) we obtain

$$\angle\left(Z_{n-1}(t+\theta_0)^*(x_\varepsilon(t)-x_0(t+\theta_0)), M^\perp(t,\theta_0)\right)$$

$$= \angle\left(Z_{n-1}(t+\theta_0)^* \frac{x_\varepsilon(t)-x_0(t+\theta_0)}{\|x_\varepsilon - x_0(\cdot+\theta_0)\|_{[0,T]}}, M^\perp(t,\theta_0)\right)$$

$$-\angle\left(Z_{n-1}(t+\theta_0)^* \frac{x_\varepsilon(t)-x_0(t+\theta_0)}{\|x_\varepsilon - x_0(\cdot+\theta_0)\|_{[0,T]}} - \frac{q_\varepsilon(t)}{\|x_\varepsilon - x_0(\cdot+\theta_0)\|_{[0,T]}}, M^\perp(t,\theta_0)\right)$$

$$+\angle\left(\left(e^{\Lambda T}\right)^* \left(\left(e^{\Lambda T}\right)^* - I\right)^{-1} \int_{t-T}^{t} Z_{n-1}(\tau+\theta_0)^* g(\tau, x_\varepsilon(\tau), \varepsilon)d\tau, M^\perp(t,\theta_0)\right).$$

But the difference of the first two terms in the right hand part of the last equality tends to zero as $\varepsilon \to 0$ and thus the thesis follows.

□

Next theorem is a reformulation of Theorem 2.1 suitable for our further considerations.

THEOREM 2.2. *Let x_0 be a nondegenerate T-periodic cycle of (1.2). Let $\{x_{\varepsilon,\lambda}\}_{\lambda\in\Lambda}$ be a family of T-periodic solutions of (1.1) such that $x_{\varepsilon,\lambda}(t) \to x_0(t+\theta_0)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0,T]$ and $\lambda \in \Lambda$. Let $l \in \mathbf{R}^n$ be an arbitrary vector such that $\langle l, \dot{x}_0(\theta_0)\rangle = 0$. Assume that $\left\langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\right\rangle \neq 0$. Then there exists $\varepsilon_0 > 0$ such that*

$$\langle l, x_{\varepsilon,\lambda}(0) - x_0(\theta_0)\rangle > 0 \quad \textit{or} \quad \langle l, x_{\varepsilon,\lambda}(0) - x_0(\theta_0)\rangle < 0$$

according as

$$\left\langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\right\rangle > 0 \quad \textit{or} \quad \left\langle l, Y_{n-1}(\theta_0)M^\perp(0,\theta_0)\right\rangle < 0$$

for any $\lambda \in \Lambda$ and any $\varepsilon \in (0,\varepsilon_0]$.

Proof. By Perron's lemma [12] (see also Demidovich ([3], Sec. III, §12) we have

$$v = Y_{n-1}(t)Z_{n-1}(t)^*v + \dot{x}_0(t)z_0(t)^*v$$

for any $v \in \mathbf{R}^n$ and $t \in \mathbf{R}$. Therefore

$$\begin{aligned}&\langle l, x_{\varepsilon,\lambda}(0) - x_0(\theta_0)\rangle\\ &= \langle l, Y_{n-1}(\theta_0)Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\\ &\quad + \dot{x}_0(\theta_0)z_0(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle\\ &\langle Y_{n-1}(\theta_0)^*l, Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle\,.\end{aligned}$$

Since $\left\langle Y_{n-1}(\theta_0)^*l, M^\perp(0,\theta_0)\right\rangle \neq 0$ then by Theorem 2.1 there exists $\varepsilon_0 > 0$ such that

$$\operatorname{sign}\langle Y_{n-1}(\theta_0)^*l, Z_{n-1}(\theta_0)^*(x_{\varepsilon,\lambda}(0) - x_0(\theta_0))\rangle = \operatorname{sign}\left\langle Y_{n-1}(\theta_0)^*l, M^\perp(0,\theta_0)\right\rangle$$

for any $\varepsilon \in (0,\varepsilon_0]$ and $\lambda \in \Lambda$ and thus the proof is complete. □

3. The topological degree of the perturbed Poincaré-Andronov operator

To proceed to the proof of our main Theorem 3.1 we need three additional theorems which are formulated below for the convenience of the reader.

Malkin's Theorem (see [9], p. 41) *Assume that T-periodic solutions x_ε of (1.1) satisfy the property $x_\varepsilon(t) \to x_0(t+\theta_0)$ as $\varepsilon \to 0$. Then $M(\theta_0) = 0$.*

Capietto-Mawhin-Zanolin Theorem (see [2], Corollary 2). *Let $V \subset \mathbf{R}^n$ be an open bounded set. Assume that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial V$. Then $d(I - \mathcal{P}_0, V) = (-1)^n d(f,V)$.*

Kamenskii-Makarenkov-Nistri Theorem (see [5], Corollary 2.8). *Assume that $\theta_0 \in [0,T]$ is an isolated zero of the bifurcation function M. Then there exist $\varepsilon_0 > 0$ and $r > 0$ such that $\mathcal{P}_\varepsilon(v) \neq v$ for any $\|v - v_0\| = r$ and any $\varepsilon \in (0, \varepsilon_0]$. Moreover $d(I - \mathcal{P}_\varepsilon, B_r(v_0)) = \mathrm{ind}(\theta_0, M)$.*

We will say that the set $U \subset \mathbf{R}^n$ has a smooth boundary if given any $v \in \partial U$ there exists $r > 0$ and a homeomorphism of $\{\xi \in \mathbf{R}^{n-1} : \|\xi\| \leq 1\}$ onto $\partial U \cap B_r(v)$. Thus any set U with a smooth boundary possesses a tangent plane to ∂U at any $v \in \partial U$. This tangent plane will be denoted by $L_U(v)$. Moreover, if U has a smooth boundary and $\mathbf{R}^n \ni h \notin L_U(v)$ then there exists $\lambda_0 > 0$ such that either $\lambda h + v \in U$ for any $\lambda \in (0, \lambda_0]$ or $\lambda h + v \notin U$ for any $\lambda \in (0, \lambda_0]$. In this case we will say that h centered at v is directed inward to U or outward respectively.

THEOREM 3.1. *Let x_0 be a nondegenerate T-periodic cycle of (1.2). Let $U \subset \mathbf{R}^n$ be an open bounded set with a smooth boundary and $x_0([0,T]) \subset \partial U$. Assume that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial U \backslash x_0([0,T])$. Assume that M has a finite number of zeros $0 \leq \theta_1 < \theta_2 < ... < \theta_k < T$ on $[0,T]$ and $\mathrm{ind}(\theta_i, M) \neq 0$ for any $i \in \overline{1,k}$. Assume that $Y_{n-1}(\theta_i) M^\perp(0, \theta_i) \notin L_U(x_0(\theta_i))$ for any $i \in \overline{1,k}$. Then for any $\varepsilon > 0$ sufficiently small $d(I - \mathcal{P}_\varepsilon, U)$ is defined. Moreover,*

$$d(I - \mathcal{P}_\varepsilon, U) = (-1)^n d(f, U) - \sum_{i=1}^{k} \mathrm{ind}(\theta_i, M) D_i,$$

where $D_i = 1$ or $D_i = 0$ according as $Y_{n-1}(\theta_i) M^\perp(0, \theta_i)$ centered at $x_0(\theta_i)$ is directed inward to U or outward.

Proof. By Kamenskii-Makarenkov-Nistri theorem there exists $r > 0$ and $\varepsilon_0 > 0$ such that

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i))) = \mathrm{ind}(\theta_i, M) \tag{3.1}$$

for any $\varepsilon \in (0, \varepsilon_0]$ and $i \in \overline{1,k}$. From Malkin's theorem we have the following "Malkin's property": $r > 0$ can be decreased, if necessary, in such a way that there exists $\varepsilon_0 > 0$ such that any T-periodic solution x_ε of (1.1) with initial condition $x_\varepsilon(0) \in B_r(x_0([0,T]))$ and $\varepsilon \in (0, \varepsilon_0]$ satisfies $x_\varepsilon(0) \in \cup_{i \in \overline{1,k}} B_r(x_0(\theta_i))$. Malkin's property implies that

$$d\left(I - \mathcal{P}_\varepsilon, \left(B_r(x_0([0,T])) \backslash \cup_{i \in \overline{1,k}} B_r(x_0(\theta_i))\right) \cap U\right) = 0 \tag{3.2}$$

for any $\varepsilon \in (0, \varepsilon_0]$. Denote by l_i the perpendicular to $L_U(x_0(\theta_i))$ directed outward away from U or inward according as $(Z_{n-1}(\theta_i)^*)^{-1} M^\perp(0, \theta_i)$ centered at $x_0(\theta_i)$ is directed outward away from U or inward. From Theorem 2.2 and Malkin's

property we have that $\varepsilon_0 > 0$ can be diminished in such a way that for any $i \in \overline{1,k}$ any T-periodic solution x_ε of (1.1) with initial condition $x_\varepsilon(0) \in B_r(x_0(\theta_i))$ and $\varepsilon \in (0, \varepsilon_0]$ satisfies $x_\varepsilon(0) \in B_r(x_0(\theta_i)) \cap U$ or $x_\varepsilon(0) \notin B_r(x_0(\theta_i)) \cap U$ according as $D_i = 1$ or $D_i = 0$. This observation allows to deduce from (3.1) that

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i)) \cap U) = \mathrm{ind}(\theta_i, M), \quad \text{if } D(\theta_i) = 1, \tag{3.3}$$

$$d(I - \mathcal{P}_\varepsilon, B_r(x_0(\theta_i)) \cap U) = 0, \quad \text{if } D(\theta_i) = 0, \tag{3.4}$$

for any $\varepsilon \in (0, \varepsilon_0]$ and $i \in \overline{1,k}$.

Observe that our choice of $r > 0$ ensures that $\mathcal{P}_0(v) \neq v$ for any $v \in \partial\left(U \backslash B_r(x_0([0,T]))\right)$. Thus, by Capietto-Mawhin-Zanolin theorem we have $d(I - \mathcal{P}_0, U \backslash B_r(x_0([0,T]))) = (-1)^n d(f, U \backslash B_r(x_0([0,T])))$. Without loss of generality we can consider $r > 0$ sufficiently small such that $d(f, U \backslash B_r(x_0([0,T]))) = d(f,U)$ obtaining

$$d(I - \mathcal{P}_0, U \backslash B_r(x_0([0,T]))) = (-1)^n d(f,U). \tag{3.5}$$

Since

$$\begin{aligned} d(I - \mathcal{P}_\varepsilon, U) = d\Big(I - \mathcal{P}_\varepsilon, \Big(B_r(x_0([0,T])) \backslash \cup_{i \in \overline{1,k}} B_r(x_0(\theta_i))\Big) \cap U\Big) \\ + d\Big(I - \mathcal{P}_\varepsilon, \cup_{i \in \overline{1,k}} B_r(x_0(\theta_i)) \cap U\Big) \\ + d(I - \mathcal{P}_\varepsilon, U \backslash B_r(x_0([0,T]))) \end{aligned}$$

the conclusion follows from formulas (3.2)-(3.5). $\square$

4. A proof of the Mawhin's conjecture

In this section we assume that the set $U \subset \mathbf{R}^n$ has a smooth boundary and there exists $v_{n-1} \in \mathbf{R}^{n-1}$ satisfying the following assumption

$$Y_{n-1}(t)\left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \left(\mathrm{e}^{\Lambda t}\right)^* v_{n-1} \notin L_U(t) \quad \text{for any } t \in [0,T]. \tag{4.1}$$

We note that assumption (4.1) does not depend on the perturbation term of (1.1) and relies to unperturbed system (1.2). Let $D = 1$ or $D = 0$ according as $Y_{n-1}(0)\left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \left(\mathrm{e}^{\Lambda t}\right)^* v_{n-1}$ centered at $x_0(0)$ is directed inward to U or outward. Given odd $m \in \mathbf{N}$ we construct the perturbation term g in such a way that $d(I - \mathcal{P}_\varepsilon, U) = (-1)^n d(f,U) - m(2D-1)$ for any $\varepsilon > 0$ sufficiently small. Without loss of generality we consider $T = 2\pi$.

Since $(z_0(t), Z_{n-1}(t))$ is nonsingular then $((z_0(t), \Phi(t))^*$ is nonsingular as well. Define $\Omega : x_0([0,2\pi]) \to \mathbf{R}^n$ as $\Omega(x_0(t)) = ((z_0(t), \Phi(t))^*)^{-1}$ for any $t \in [0, 2\pi]$. By Uryson's theorem (see [6], Ch. 1, Theorem 1.1) Ω can be continued to the whole $\mathbf{R}^n$ in such a way that $\Omega \in C^0(\mathbf{R}^n, \mathbf{R}^n)$. Analogously, we consider

$\widetilde{\Gamma} \in C^0(\mathbf{R}^n, \mathbf{R}^n)$ such that $\widetilde{\Gamma}(x_0(t)) = (\arcsin(\sin t), 0, \ldots, 0)^*$ and denote by $\Gamma \in C^0(\mathbf{R}^n, \mathbf{R})$ the first component of $\widetilde{\Gamma}$. Let us define a 2π-periodic α-approximation of $\left((\mathrm{e}^{\Lambda t})^*\right)^{-1}$ on $[-2\pi, 0]$ by

$$\mathrm{e}_\alpha(t) = ((\mathrm{e}^{\Lambda t})^*)^{-1}, \quad \text{if } t \in [-2\pi, -\alpha],$$

$$\mathrm{e}_\alpha(t) = \frac{t}{-\alpha}\left((\mathrm{e}^{-\Lambda\alpha})^*\right)^{-1} + \left(1 - \frac{t}{-\alpha}\right)\left((\mathrm{e}^{-2\pi\Lambda})^*\right)^{-1}, \quad \text{if } t \in [-\alpha, 0],$$

which is continued to $(-\infty, \infty)$ by the 2π-periodicity. We are now in a position to introduce the required perturbation term, namely we consider that the perturbed system (1.1) has the following form

$$\dot{x} = f(x) + \varepsilon\Gamma(x)\Omega(x)\begin{pmatrix} D\sin(mt) + (1-D)\cos(mt) \\ (D\cos(mt) + (1-D)\sin(mt))\mathrm{e}_\alpha(t)v_{n-1}\end{pmatrix}, \tag{4.2}$$

where $\alpha > 0$ is sufficiently small. Consequently we denote by $\mathcal{P}_\varepsilon$ the Poincaré-Andronov operator of system (4.2) over the period 2π.

Proposition 4.1. *Let $x_0([0,T]) \subset U \subset \mathbf{R}^n$ be an open bounded set with a smooth boundary and assume that there exists $v_{n-1} \in \mathbf{R}^n$ such that (4.1) is satisfied. Then given any odd $m > 0$ there exists $\alpha_0 > 0$ such that for any fixed $\alpha \in (0, \alpha_0]$ and $\varepsilon > 0$ sufficiently small $d(I - \mathcal{P}_\varepsilon, U)$ is defined and*

$$d(I - \mathcal{P}_\varepsilon, U) = \begin{cases} (-1)^n d(f,U) - m, & \text{if } D = 1, \\ (-1)^n d(f,U) + m, & \text{if } D = 0. \end{cases}$$

Proof. By the definition of Ω and Γ we have

$$\begin{pmatrix} z_0(t)^* \\ Z_{n-1}(t)^* \end{pmatrix}\Omega(x_0(t)) = \begin{pmatrix} 1 & 0 \\ 0 & (\mathrm{e}^{\Lambda t})^* \end{pmatrix}, \tag{4.3}$$

$$\Gamma(x_0(t)) = \arcsin(\sin t).$$

Therefore, taking into account that m is odd, we obtain the following formula for the bifurcation function M

$$\begin{aligned} M(\theta) &= \int_0^{2\pi} \arcsin(\sin\tau)(D\sin(m(\tau-\theta)) + (1-D)\cos(m(\tau-\theta)))d\tau \\ &= (-1)^{(m-1)/2}\frac{4D\cos(m\theta) + 4(1-D)\sin(m\theta)}{m^2} \end{aligned}$$

whose zeros are $\theta_j = \frac{1}{m}\left(\frac{D\pi}{2} + j\pi\right)$, $j \in \overline{0, 2m-1}$. Moreover,

$$\operatorname{ind}(\theta_j, M) = \operatorname{sign}(M'(\theta_j)) \tag{4.4}$$

$$= (-1)^{(m-1)/2}\operatorname{sign}\left(\frac{4m\left(-D\sin\left(D\frac{\pi}{2} + j\pi\right) + (1-D)\cos\left(D\frac{\pi}{2} + j\pi\right)\right)}{m^2}\right).$$

Let us denote by $M_\alpha^\perp$ the function $M^\perp$ corresponding to system (4.2). From (4.3) we have that

$$M_\alpha^\perp(0,\theta) = \left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \int\limits_{-2\pi}^{0} (Z_{n-1}(s+\theta))^* g(s, x_0(s+\theta), 0)ds$$

$$= \left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \left(\mathrm{e}^{\Lambda \theta}\right)^* \circ$$

$$\circ \int\limits_{-2\pi}^{0} \left(\mathrm{e}^{\Lambda s}\right)^* \mathrm{e}_\alpha(s) v_{n-1} \arcsin(\sin(s+\theta))(D\cos(ms) + (1-D)\sin(ms))ds.$$

Since

$$\int\limits_{-2\pi}^{0} \arcsin(\sin(s+\theta))(D\cos(ms) + (1-D)\sin(ms))ds$$

$$= -(-1)^{(m-1)/2} \cdot \frac{4(D\sin(m\theta) + (1-D)\cos(m\theta))}{m^2}$$

by taking into account that m is odd we have that $M_\alpha^\perp(0,\theta) \to M_0^\perp(0,\theta)$ as $\alpha \to 0$, where

$$M_0^\perp(0,\theta) = -\left(\mathrm{e}^{\Lambda T}\right)^* \left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1} \left(\mathrm{e}^{\Lambda \theta}\right)^* v_{n-1}(-1)^{(m-1)/2}$$

$$\cdot \frac{4(D\sin(m\theta) + (1-D)\cos(m\theta))}{m^2}.$$

Put $q(\theta) = -(-1)^{(m-1)/2}(D\sin(m\theta) + (1-D)\cos(m\theta))$. Then, taking any $\theta \in [0, 2\pi]$ and using the definition of D we conclude that $Y_{n-1}(\theta)M_0^\perp(0,\theta)$ centered at $x_0(\theta)$ is directed inward to U or outward according as $\mathrm{sign}(q(\theta))(2D-1) = 1$ or $\mathrm{sign}(q(\theta))(2D-1) = -1$. Therefore, there exists $\alpha_0 > 0$ such that for any $\alpha \in [0, \alpha_0]$ and any $\theta \in [0, 2\pi]$ we have that $Y_{n-1}(\theta)M_\alpha^\perp(0,\theta)$ centered at $x_0(\theta)$ is directed inward to U or outward according as $\mathrm{sign}(q(\theta))(2D-1) = 1$ or $\mathrm{sign}(q(\theta))(2D-1) = -1$. Thus denoting by $\mathcal{P}_{\varepsilon,\alpha}$ the Poincaré-Andronov operator of system (4.2) from Theorem 3.1 we have that

$$d(I - \mathcal{P}_{\varepsilon,\alpha}, U) = (-1)^n d(f, U) - \sum_{j \in \overline{0, 2m-1} : \mathrm{sign}(q(\theta_j))(2D-1) = 1} \mathrm{ind}(\theta_j, M) \tag{4.5}$$

for any $\alpha \in (0, \alpha_0]$. Consider the case when $D = 1$. Then the property $\mathrm{sign}(q(\theta_j))(2D-1) = 1$ is equivalent to

$$(-1)^{(m-1)/2}\mathrm{sign}(\sin(\pi/2 + j\pi)) = -1. \tag{4.6}$$

If $j \in \overline{0, 2m-1}$ satisfies (4.6) then (4.4) implies $\mathrm{ind}(\theta_j, M) = 1$. Since there exists exactly m elements of $\overline{0, 2m-1}$ satisfying (4.6) then (4.5) can be rewritten as $d(I - \mathcal{P}_\varepsilon, U) = d(f, U) - m$. Analogously, if $D = 0$ then $\mathrm{sign}(q(\theta_j))(2D-1) = 1$

is equivalent to $(-1)^{(m-1)/2}\mathrm{sign}(\cos(j\pi)) = -1$ that in combination with (4.4) gives $\mathrm{ind}(\theta_j, M) = -1$ allowing to rewrite (4.5) in the form $d(I - \mathcal{P}_\varepsilon, U) = d(f, U) + m$.

$\square$

At the end of the paper we note that system (1.2) should exhibit very complex behavior in order that assumption (4.1) be not satisfied with any $v_{n-1} \in \mathbf{R}^{n-1}$. Particularly, (4.1) holds true for the prototypic unperturbed system (1.2)

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1), \\ \dot{x}_2 &= -x_1 - x_2(x_1^2 - x_2^2 - 1), \\ \dot{x}_3 &= -x_3 \end{aligned} \tag{4.8}$$

possessing the nondegenerate 2π-periodic cycle $x_0(t) = \binom{\sin t}{\cos t}$ and $U = B_1(0) = \{v \in \mathbf{R}^3 : \|v\| < 1\}$. Indeed, it can be easily checked that $\Phi(t) = \left(\binom{\sin t}{0}, \binom{\cos t}{0}, \binom{0}{1}\right)^*$, $\mathrm{e}^{\Lambda t} = \begin{pmatrix} \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix}$ and $Y_{n-1}(t) = \Phi(t)\mathrm{e}^{-\Lambda t}$ in this case. Thus, taking $v_{n-1} = \binom{1}{0}$ we have

$$Y_{n-1}(t)\left(\mathrm{e}^{\Lambda T}\right)^*\left(\left(\mathrm{e}^{\Lambda T}\right)^* - I\right)^{-1}\left(\mathrm{e}^{\Lambda t}\right)^* v_{n-1} = \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}-1}\left(\sin t, \cos t, 0\right)^*.$$

This last vector centered at $x_0(t)$ is perpendicular to ∂U for any $t \in [0, 2\pi]$.

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